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CHARACTER OF THE OSCILLATOR REPRESENTATION

BY

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ABSTRACT

We compute the character of the oscillator representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. The proof, which uses coherent continuation, is completely algebraic. We show that the character of the difference of the two halves of the oscillator representation is the quotient of Weyl denominators of type B_n and C_n, and thus has the form of a transfer factor for $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}(2n+1)$.

1. Introduction

The oscillator representation ω of the metaplectic group $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is the smallest representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ which is genuine, i.e. does not factor to $\mathrm{Sp}(2n,\mathbb{R})$. The main result of this paper is a short computation of the global character of ω. Formally the argument is very simple and we sketch it in the introduction. We will be more precise starting in Section 2.

Let π_{DS} be the holomorphic discrete series representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ with the same infinitesimal character as that of the oscillator representation. Let $\omega=\omega_{\mathrm{even}}\oplus\omega_{\mathrm{odd}}$ be the decomposition of ω into irreducible representations, and write $\omega_{\mathrm{even}}-\omega_{\mathrm{odd}}$ for the difference in the Grothendieck group.

It follows from general principles that

$$\omega_{\mathrm{even}}-\omega_{\mathrm{odd}}=\frac{(-1)^q}{n!}\sum_{w\in W} w\cdot\pi_{DS}. \tag{1.1}$$

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Here $q = \frac{1}{2}\dim(\widetilde{\mathrm{Sp}}(2n,\mathbb{R})/K) = n(n+1)/2$, where K is a maximal compact subgroup. Also W denotes the complex Weyl group of type C_n, and $\pi \to w\cdot\pi$ is the coherent continuation action of W on virtual modules.

The global character Θ_{DS} of π_{DS} is known explicitly as a function on the regular elements [5], [10]:

$$\Theta_{DS} = \pm\frac{\sum_{w\in W_K}\mathrm{sgn}(w)e^{w\lambda}}{\prod_\alpha(e^{\alpha/2}-e^{-\alpha/2})}. \tag{1.2}$$

Here W_K is the Weyl group of K, λ is the Harish-Chandra parameter of π_{DS} and the product in the denominator is over the positive roots Δ^+ defined by λ. This is Harish-Chandra's formula on the compact Cartan subgroup, and appropriately interpreted holds on any Cartan subgroup by virtue of the fact that π_{DS} is a holomorphic representation.

Let $\theta_{\text{even,odd}}$ be the global character of $\omega_{\text{even,odd}}$, as a function on the regular set. It follows immediately from (1.1), (1.2) and the definition of coherent continuation that

$$\theta_{\text{even}} - \theta_{\text{odd}} = \frac{\pm\sum_{w\in W}\mathrm{sgn}(w)e^{w\lambda}}{\prod_\alpha(e^{\alpha/2}-e^{-\alpha/2})} \tag{1.3}$$

(the precise statement is Theorem 3.11). Furthermore $\lambda = (n-\frac{1}{2}, n-\frac{3}{2},\dots,\frac{1}{2})$ and, formally at least, this is one-half the sum of the positive roots for the root system of type B_n. Therefore by the usual product formula the numerator of (1.3) equals

$$\prod_{\alpha\in\Delta^+(B_n)}(e^{\alpha/2}-e^{-\alpha/2}). \tag{1.4}$$

Write $\Delta^+(C_n)$ for the positive roots appearing in (1.2). The long roots of $\Delta^+(B_n)$ are the same as the short roots of $\Delta^+(C_n)$, whereas the short roots of $\Delta^+(B_n)$ are one-half times the long roots $\{\alpha_1,\dots,\alpha_n\}$ of $\Delta^+(C_n)$. Thus

$$\text{(1.5)(a)}\qquad \theta_{\text{even}} - \theta_{\text{odd}} = \frac{\pm\prod_{\Delta^+(B_n)}(e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\Delta^+(C_n)}(e^{\alpha/2}-e^{-\alpha/2})}$$

$$\text{(b)}\qquad = \frac{\pm 1}{\prod_{j=1}^n(e^{\alpha_j/4}+e^{-\alpha_j/4})}.$$

Now suppose (for simplicity) that G is a connected complex reductive group, and g is a semisimple element of G. Let π be a finite dimensional representation of G such that for every weight α of π, $-\alpha$ is also a weight. Write the weights of π as a disjoint union $S \cup -S$. Then

$$\begin{aligned}\det(1+\pi(g)) &= \prod_{\alpha\in S}(1+e^{\alpha}(g))(1+e^{-\alpha}(g))\\ &= \prod_{\alpha\in S}(e^{-\alpha/2}+e^{\alpha/2})(e^{\alpha/2}+e^{-\alpha/2})(g)\\ &= \prod_{\alpha\in S}(e^{\alpha/2}+e^{-\alpha/2})^2(g).\end{aligned} \tag{1.6}$$

The weights of the standard representation of $\mathrm{Sp}(2n,\mathbb{R})$ are $\pm\alpha_1/2,\dots,\pm\alpha_n/2$. This observation applied to (1.5)(b) gives

$$(\theta_{\text{even}}-\theta_{\text{odd}})(\tilde g)^2 = \frac{1}{\det(1+g)}$$

where g is the image of $\tilde g$ in $\mathrm{Sp}(2n,\mathbb{R})$. Therefore

$$(\theta_{\text{even}}-\theta_{\text{odd}})(\tilde g) = \frac{c(\tilde g)}{|\det(1+g)|^{\frac12}} \tag{1.7}$$

with $c(\tilde g)^4 = 1$.

With a little extra work, mostly having to do with the covering group, these formal arguments may be made precise, and the constants computed (Theorem 3.11 and Proposition 4.5).

We also describe an alternative formulation. We write a Cartan subgroup $\tilde H$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ in terms of a certain cocycle, closely related to the Rao cocycle [11], so an element of $\tilde H$ is a pair $(g;\epsilon)$ with $g\in H$ and $\epsilon=\pm1$. Then (1.7) takes the simple form (Proposition 5.11)

$$(\theta_{\text{even}}-\theta_{\text{odd}})(g;\epsilon) = \frac{\epsilon}{\sqrt{\det(1+g)}}. \tag{1.8}$$

Actually the oscillator representation depends on a choice of an additive character of $\mathbb{R}$, which enters on the right hand side of (1.8) in the choice of a branch of the square root.

Finally ω_{even} and ω_{odd} have distinct central characters. Evaluating (1.3) and (1.7) at an element $z\tilde g$, where z is a central element lying over the element $-I$ of $\mathrm{Sp}(2n,\mathbb{R})$, gives formulas (Propositions 3.18 and 4.7) for the global character of $\omega=\omega_{\text{even}}\oplus\omega_{\text{odd}}$. However, it is interesting to note the analogue of (1.8) has a more complicated numerator; similarly the analogue of (1.3) is less natural.

The character of the oscillator representation has been computed in ([14] ,§I, Théorème 2), using very different methods. Statement (1.7) (for $\omega_{\text{even}} \oplus \omega_{\text{odd}}$) appears in [7] (without the computation of $c(\tilde{g})$). This method of proof may be applied to compute characters of small representations of other groups.

In some sense (1.5)(a) is the main result of this paper. We write this as

$$\theta_{\text{even}} - \theta_{\text{odd}} = \frac{D_{\text{SO}}}{D_{\text{Sp}}} \tag{1.9}$$

where $D_{\text{Sp}}, D_{\text{SO}}$ are Weyl denominators for $\text{Sp}(2n,\mathbb{R})$ and $\text{SO}(2n+1)$ respectively. Thus $\theta_{\text{even}} - \theta_{\text{odd}}$ has the form of a *transfer factor* [9] between characters of $\widetilde{\text{Sp}}(2n,\mathbb{R})$ and $\text{SO}(2n+1)$. The formulas of this paper have been written with this application in mind, and we give a few details.

There is a bijection between stable, (strongly) regular semisimple conjugacy classes of $\text{SO}(n+1,n)$ and $\text{Sp}(2n,\mathbb{R})$ — two such conjugacy classes correspond if they have the same non-trivial eigenvalues. Equivalently, if H is a Cartan subgroup of $\text{SO}(n+1,n)$ we may choose an isomorphism ϕ: $H \to H'$ with a Cartan subgroup of $\text{Sp}(2n,\mathbb{R})$. Let Θ be a stable invariant eigendistribution on $\text{SO}(n+1,n)$, identified with a function on the (strongly) regular semisimple elements.

We define a function Θ' on $\widetilde{\text{Sp}}(2n,\mathbb{R})$ by

$$\Theta'(g') = \Theta(g)(\theta_{\text{even}} - \theta_{\text{odd}})(g') \quad (g' \in \widetilde{\text{Sp}}(2n,\mathbb{R}))$$

where g corresponds to the image $p(g') \in \text{Sp}(2n,\mathbb{R})$ of g'. This is independent of the choice of g since Θ is stable.

Roughly speaking, using the identification of H and H', a stable invariant eigendistribution for $\text{SO}(n+1,n)$, when multiplied by $\theta_{\text{even}} - \theta_{\text{odd}}$, is a candidate for an invariant eigendistribution on $\widetilde{\text{Sp}}(2n,\mathbb{R})$. This follows from Harish-Chandra's formula for the restriction of an invariant eigendistribution to a Cartan subgroup [4]: Θ and Θ' have the same "numerator" on $H \simeq H'$. This can be made precise, and the map $\Theta \to \Theta'$ is a bijection between stable invariant eigendistributions on $\text{SO}(n+1,n)$ and genuine, stable invariant eigendistributions on $\widetilde{\text{Sp}}(2n,\mathbb{R})$. This application is discussed elsewhere [1].

2. Coherent continuation

We fix once and for all a real vector space W of dimension $2n$ equipped with a non-degenerate symplectic form $<,>$, and we fix a standard basis $e_1, \ldots, f_n$ of W. The isometry group of $<,>$ is then $\mathrm{Sp}(2n,\mathbb{R})$ realized as the set of $2n \times 2n$ real matrices satisfying $gJ\,{}^tg^{-1} = J$, where $J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. The real and complex Lie algebras are denoted $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{sp}(2n,\mathbb{C})$ respectively, realized as real or complex matrices X satisfying $XJ + J\,{}^tX = 0$. Let $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ be the two-fold cover of $\mathrm{Sp}(2n,\mathbb{R})$, with covering map p: $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$. It is well known that $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is unique up to isomorphism. In Section 5 we will choose a particular model of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, but until then this will not be necessary.

Let $\tilde{K}$ be the inverse image in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ of the usual maximal compact subgroup $K = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{SO}(2n,\mathbb{R}) \simeq U(n)$ of $\mathrm{Sp}(2n,\mathbb{R})$. Unless otherwise noted by "module" we mean admissible $(\mathfrak{sp}(2n,\mathbb{C}),\tilde{K})$-module, and by virtual module we mean an element of the corresponding Grothendieck group, i.e. a formal finite linear combination of irreducible modules with integral coefficents.

Given a non-trivial unitary additive character ψ of $\mathbb{R}$ the associated oscillator representation $\omega(\psi)$ is the direct sum $\omega(\psi) = \omega(\psi)_{\mathrm{even}} \oplus \omega(\psi)_{\mathrm{odd}}$, where $\omega(\psi)_{\mathrm{even}}$ and $\omega(\psi)_{\mathrm{odd}}$ are irreducible unitary representations and $\omega(\psi)_{\mathrm{even}}$ contains a $\tilde{K}$-invariant line. (In the usual model $\omega(\psi)_{\mathrm{even,odd}}$ is realized on a space of even (odd) functions, respectively.) According to the Hermitian symmetric structure write $\mathfrak{sp}(2n,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ as usual. Define $\psi_\pm(x) = e^{\pm 2\pi i x}$. We label $\mathfrak{p}^\pm$ so that $\omega(\psi_\pm)_{\mathrm{even,odd}}$ have cyclic vectors under the action of $\mathfrak{p}^\pm$, equivalently a vector annihilated by $\mathfrak{p}^\mp$. We reserve the term "highest weight module" for a module with a vector annihilated by $\mathfrak{p}^-$.

For $a_1, \ldots, a_n \in \mathbb{C}$ let $(a_1, \ldots, a_n) = \begin{pmatrix} & X \\ -X & \end{pmatrix}$ with $X = \mathrm{diag}(a_1, \ldots, a_n)$. The set of such elements with $a_i \in \mathbb{R}$ (resp. $\mathbb{C}$) forms the real (resp. complexified) Lie algebra $\mathfrak{t}_0$ (resp. $\mathfrak{t}$) of a compact Cartan subgroup T of $\mathrm{Sp}(2n,\mathbb{R})$. For H a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ we let $\tilde{H}$ denote its inverse image in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. The compact Cartan subgroup $\tilde{T}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is connected and abelian (cf. Section 3). We write $(a_1, \ldots, a_n)$ for the element of $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t},\mathbb{C})$ taking $(z_1, \ldots, z_n)$ to $i \sum a_j z_j$. Then the set of roots of $\mathfrak{t}$ in $\mathfrak{sp}(2n,\mathbb{C})$ is $\{\pm e_i \pm e_j, \pm 2e_i\}$ as usual. The weight lattice of $\mathfrak{t}$ in $\mathfrak{sp}(2n,\mathbb{C})$ is $\Lambda = \{(a_1, \ldots, a_n) \,|\, a_i \in \mathbb{Z}\}$. Let $\Delta_c^+ = \{e_i - e_j | i < j\}$ and $\Delta_n^+ = \{2e_i, e_i + e_j\}$, the positive compact and non-compact roots, and let $\Delta^+ = \Delta_n^+ \cup \Delta_c^+$. Write ρ_c, ρ_n and ρ for one-half the sum of the positive roots in these sets respectively. The finite-dimensional irreducible representations of $\tilde{K}$ are parametrized by highest weights $(a_1,\dots,a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_i \in \mathbb{Z}$ for all i or $a_i \in \mathbb{Z}+\frac{1}{2}$ for all i.

The infinitesimal character of $\omega(\psi)$ is $\lambda_0 = (n-\frac{1}{2}, n-\frac{3}{2},\dots,\frac{1}{2})$ (identified via the Harish-Chandra homomorphism). We work entirely in the category of representations with this infinitesimal character.

Definition 2.1: Let π be a virtual module with infinitesimal character λ_0. Let Φ be a coherent family based on $\tilde{T}$ and

$$\lambda_0 + \Lambda = \{(a_1,\dots,a_n) \mid a_i \in \mathbb{Z}+\tfrac{1}{2}\}$$

([15], Definition 7.2.5 and Corollary 7.2.27). For $w \in W = W(\mathfrak{sp}(2n,\mathbb{C}),\mathfrak{t})$ define

$$w\cdot\pi = \Phi(w^{-1}\lambda_0)$$

([15], Definition 7.2.28). This is a virtual module with infinitesimal character λ_0.

A few comments are in order since we are not precisely in the setting of [15]. Since $\tilde{T}$ is connected we freely identify the formal symbols $\lambda_0+\Lambda$ of ([15], Definition 7.2.5) with elements of $\mathfrak{t}^*$. Although $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is not a linear group, ([15], Corollary 7.2.27) carries over without any changes. The integral roots $\Delta(\lambda_0)$ and integral Weyl group $W(\lambda_0)$ of λ_0 are of type D_n; however ([15], Lemma 7.2.29) is easily seen to hold for the larger set $\{w \in W \mid w\lambda_0 - \lambda \text{ is a sum of weights}\}$ which in our case is all of W.

We note that the more detailed results on wall-crossing of ([15], §7) fail with W in place of $W(\lambda_0)$ (cf. Remark 2.10).

Now let π_{DS} be the holomorphic discrete series representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ with infinitesimal character λ_0. This has lowest K-type $(n+\frac{1}{2},\dots,n+\frac{1}{2})$, and is a highest weight module. Let

$$\hat{\pi}_{DS} = \sum_{w\in W} \mathrm{sgn}(w) w\cdot\pi_{DS}. \tag{2.2}$$

THEOREM 2.3:

$$\hat{\pi}_{DS} = (-1)^q n! [\omega(\psi_+)_{\mathrm{even}} - \omega(\psi_+)_{\mathrm{odd}}].$$

Proof: It follows immediately from the definition that all simple roots α of $\Delta^+(\lambda_0) = \Delta^+ \cap \Delta(\lambda_0)$ are in the τ-invariant [15] of $\hat{\pi}_{DS}$. Therefore by ([16], Corollary 4.7) any irreducible summand σ of $\hat{\pi}_{DS}$ has Gelfand–Kirillov dimension

n. Since the irreducible summands of a highest weight module tensored with a finite dimensional representation are all highest weight modules, this holds also for σ. By the classification of highest weight modules for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$[3], the only possibilities for σ are $\omega(\psi_+)_{\mathrm{even}}$ and $\omega(\psi_+)_{\mathrm{odd}}$. Thus

$$\hat{\pi}_{DS} = a\omega(\psi_+)_{\mathrm{even}} + b\omega(\psi_+)_{\mathrm{odd}}$$

for some integers a, b. To evaluate a, b we use the following Blattner-type formula.* Let $m(\mu, \hat{\pi}_{DS})$ denote the multiplicity of a K-type μ in $\hat{\pi}_{DS}$.

PROPOSITION 2.4: *Let μ be an irreducible representation of $\tilde{K}$ (identified with its highest weight). Let Q denote the Kostant multiplicity function for the set Δ_n^+ [8]. Then*

$$m(\mu, \hat{\pi}_{DS}) = \sum_{w\in W}\sum_{y\in W_K} \mathrm{sgn}(w)\,\mathrm{sgn}(y) Q(y(\mu+\rho_c) - (w\lambda_0 + \rho_n)).$$

This is well-known to the experts, but for lack of a reference we give a proof.

Proof: We prove the stronger statement

$$m(\mu, w\cdot\pi_{DS}) = \sum_{y\in W_K} \mathrm{sgn}(y) Q(y(\mu+\rho_c) - (w^{-1}\lambda_0 + \rho_n)). \tag{2.5}$$

Write π_{DS} as a derived functor module $\mathcal{R}_{\mathfrak{q}}^S(\lambda_0)$, also satisfying $\mathcal{R}_{\mathfrak{q}}^i(\lambda_0) = 0$ for $i \neq S$ ([17], Theorem 6.8). Here $\mathfrak{q} = \mathfrak{t}\oplus\mathfrak{u}$ is the Borel subalgebra defined by Δ^+ (i.e. making λ_0 dominant), and $S = \frac{1}{2}\dim(K/T) = n(n-1)/2$.

Thus

$$\begin{aligned} m(\mu, w\cdot\pi_{DS}) &= m(\mu, w\cdot\mathcal{R}_{\mathfrak{q}}^S(\lambda_0)) \\ &= m(\mu, w\cdot\sum_i(-1)^{S-i}\mathcal{R}_{\mathfrak{q}}^i(\lambda_0)). \end{aligned}$$

By ([15], Corollary 7.2.10), this equals

$$m(\mu, \sum_i(-1)^{S-i}\mathcal{R}_{\mathfrak{q}}^i(w^{-1}\lambda_0)).$$

Now (2.5) follows from a Blattner formula ([15], Theorem 6.3.12); see also ([18], Theorem 6.5.3).

* We thank the referee for pointing out that the proof may also be completed by considering $\hat{\pi}_{DS}$ as a Verma module, and applying standard results in this category.

It is not difficult to evaluate the sum of Proposition 2.4 for $\mu_{\text{even}} = (\frac{1}{2}, \dots, \frac{1}{2})$ and $\mu_{\text{odd}} = (\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, the lowest K-types of $\omega(\psi_+)_{\text{even,odd}}$ respectively. Since Δ_n^+ is invariant under W_K, $y\rho_n = \rho_n$ and $Q(y\tau) = Q(\tau)$ for all $y \in W_K$ and all τ. Replacing w by $y^{-1}\lambda$ and changing variables shows

$$m(\mu, \hat{\pi}_{DS}) = n! \sum_{w \in W} \operatorname{sgn}(w) Q(\mu + \rho_c - \rho_n - w\lambda_0). \tag{2.6}$$

Define w_0 by: $w_0\rho_c = \rho_c$ and $w_0\rho_n = -\rho_n$; the length of w_0 is the number of positive non-compact roots, i.e. $q = n(n+1)/2$. Let $\tau = \mu_{\text{even}} + \rho_c - \rho_n = (-\frac{1}{2}, -\frac{3}{2}, \dots, -n + \frac{1}{2})$. Then $w_0\lambda_0 = \tau$, and replacing w by ww_0 gives

$$m(\mu_{\text{even}}, \hat{\pi}_{DS}) = n!(-1)^q \sum_{w \in W} \operatorname{sgn}(w) Q(\tau - w\tau). \tag{2.7}$$

It is easy to see $Q(\tau - w\tau)$ is non-zero if and only if $w = 1$, giving $m(\mu_{\text{even}}, \hat{\pi}_{DS}) = n!(-1)^q$.

The computation for μ_{odd} is similar, with τ replaced by $\tau + (1, 0, \dots, 0)$, and w_0 replaced by $s_{2e_1}w_0$. It follows that $m(\mu_{\text{odd}}, \hat{\pi}_{DS}) = -n!(-1)^q$. This completes the proof of Theorem 2.3.

The Weyl group $W(D_n)$ of type D_n is embedded naturally in W. Let ϕ: $W \to \pm 1$ be the corresponding quotient map:

$$1 \to W(D_n) \to W \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \to 1. \tag{2.8(a)}$$

For later use we note that $\phi(w) = \operatorname{sgn}(w)\operatorname{sgn}(p(w))$ where p: $W \to W_K$ is given by

$$1 \to (\mathbb{Z}/2\mathbb{Z})^n \to W \xrightarrow{p} W_K \simeq S^n \to 1. \tag{2.8(b)}$$

The next result is a corollary of the proof of Theorem 2.3.

Corollary 2.9:

$$\sum_{\phi(w) = (-1)^n} \operatorname{sgn}(w) w \cdot \pi_{DS} = n!(-1)^q \omega(\psi_+)_{\text{even}},$$

$$\sum_{\phi(w) = (-1)^{n+1}} \operatorname{sgn}(w) w \cdot \pi_{DS} = -n!(-1)^q \omega(\psi_+)_{\text{odd}}.$$

For any σ with $\phi(\sigma) = -1$,

$$\sigma \cdot \omega(\psi_+)_{\text{even}} = \omega(\psi_+)_{\text{odd}}.$$

Dualizing, the corresponding result holds for the antiholomorphic discrete series representation π_{DS}^* and $\omega(\psi_-)_{\text{even}}, \omega(\psi_-)_{\text{odd}}$.

Remark 2.10: The root $2e_n$ is simple for Δ^+, but is not in the integral root system $\Delta(\lambda_0)$. Nevertheless the action of $\sigma = s_{2e_n}$ is defined. This is an example of "exotic" wall-crossing (though not of an integral wall). In particular σ takes the irreducible representation $\omega(\psi_+)_{\text{even}}$ to an irreducible representation $(\omega(\psi_+)_{\text{odd}})$, something which is forbidden for ordinary (integral) wall crossing ([15], Corollary 7.3.19).

Remark 2.11: Let $\hat{\pi}'_{DS} = \sum_w \operatorname{sgn}(w) w \cdot \pi'_{DS}$ where π'_{DS} is *any* discrete series representation with infinitesimal character λ_0. By Harish-Chandra's character formula the characters of $\hat{\pi}_{DS}$ and $\hat{\pi}'_{DS}$ have the same restriction to the elliptic set, and it follows that

$$\hat{\pi}'_{DS} = a[\omega(\psi_+)_{\text{even}} - \omega(\psi_+)_{\text{odd}}] + b[\omega(\psi_-)_{\text{even}} - \omega(\psi_-)_{\text{odd}}]$$

for some integers $a + b = (-1)^q n!$. However, the evaluation of a, b by means of Proposition 2.4 is not at all easy outside of the holomorphic/anti-holomorphic cases. For example, for $\mathrm{Sp}(6,\mathbb{R})$ and π'_{DS} a large discrete series representation (i.e. corresponding to a Weyl chamber with no simple compact roots), $a, b = 4, 2$ or $2, 4$.

3. Character formulas

We begin by realizing representatives for the conjugacy classes of Cartan subgroups via Cayley transforms based on T as in [12].

For S a set of strongly orthogonal non-compact roots of $\mathfrak{t}$, let $c = c^S$ be a Cayley transform associated to S. This is an element of $\mathrm{Sp}(2n,\mathbb{C})$, and $\mathfrak{h} = ad(c)(\mathfrak{t})$ is a Cartan subalgebra of $\mathfrak{sp}(2n,\mathbb{C})$. Let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{sp}(2n,\mathbb{R})$. The centralizer H of $\mathfrak{h}_0$ in $\mathrm{Sp}(2n,\mathbb{R})$ is a Cartan subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. The inverse image $\tilde{H}$ of H in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is a Cartan subgroup of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$; $\tilde{H}$ is abelian and equal to the centralizer of $\mathfrak{h}_0$ in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Every Cartan subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ or $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is conjugate to one obtained in this manner.

We now write $\Delta^+(T)$ for the positive roots Δ^+ defined in Section 1. Then $\Delta^+(H) = c^{*-1}\Delta^+(T)$ is a set of positive roots of $\mathfrak{h}$ (c^*: $\mathfrak{h}^* \to \mathfrak{t}^*$ is the adjoint of c). Then $\Delta^+(H)$ comes equipped with a set of strongly orthogonal real roots which we write

$$\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_s \tag{3.1}$$

where $\beta_1,\ldots,\beta_m$ are short and $\gamma_1,\ldots,\gamma_s$ are long.

As is well known $\tilde{K}$ is isomorphic to the cover of $K\simeq U(n)$ defined by the square root of the determinant [2]. This is the group $\{(g,z)\mid g\in U(n), z\in\mathbb{C}^*, \det(g)=z^2\}$. It follows that the compact Cartan subgroup $\tilde{T}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is connected and the exponential map $\widetilde{\exp}\colon \mathfrak{t}_0\to\tilde{T}$ is surjective with kernel

$$\{2\pi(a_1,\ldots,a_n)\in\mathfrak{t}_0 | a_i\in\mathbb{Z},\sum_i a_i\in 2\mathbb{Z}\}. \tag{3.2}$$

For $\lambda\in\lambda_0+\Lambda$ (Definition 2.1) and $t=\widetilde{\exp}(X)\in\tilde{T}$, define $e^\lambda(t)=e^{\lambda(X)}$ as usual.

For a Cayley transform c and H as above we write $H=T_HA$ with $T_H=T\cap H$ and $A=\exp(\mathfrak{a}_0=\mathfrak{h}_0\cap\mathfrak{p}_0)$. (We identify the exponential of $\mathfrak{a}_0$ in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ with $\exp(\mathfrak{a}_0)\in\mathrm{Sp}(2n,\mathbb{R})$.) Then $\tilde{H}\simeq\tilde{T}_HA$, and we may write elements of $\tilde{H}$ in the form

$$g=\widetilde{\exp}(Z)\exp(X)\quad (Z\in\mathfrak{t}_0, X\in\mathfrak{a}_0). \tag{3.3(a)}$$

For $\lambda\in\lambda_0+\Lambda\subset\mathfrak{t}^*$ and $g\in\tilde{H}$ define

$$e^\lambda(g)=e^\lambda(t)e^{c^{*-1}\lambda(X)} \tag{b}$$

where $g=t\cdot\exp(X)\in\tilde{T}_HA$. Equivalently for $\lambda\in\mathfrak{h}^*$, $c^*(\lambda)\in\lambda_0+\Lambda$ define

$$e^\lambda(g)=e^{c^*\lambda}(t)e^{\lambda(X)}. \tag{c}$$

Proposition 3.4: *Let c be a Cayley transform with corresponding Cartan subgroup $\tilde{H}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Write $\gamma_1,\ldots,\gamma_s$ for the long real roots of $\Delta^+(H)$. Let g be a regular element of $\tilde{H}$ satisfying*

$$|e^{\gamma_i}(g)|<1,\quad i=1,\ldots,s. \tag{3.5}$$

Then

$$[\theta(\psi_+)_{\text{even}}-\theta(\psi_+)_{\text{odd}}](g)=\frac{\sum_{w\in W}\mathrm{sgn}(w)e^{w\lambda_0}}{\prod_{\alpha\in\Delta^+(H)}(e^{\alpha/2}-e^{-\alpha/2})}(g). \tag{3.6}$$

Every regular element of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is conjugate to some g satisfying (3.5), so (3.6) determines $\theta(\psi_+)_{\text{even}}-\theta(\psi_+)_{\text{odd}}$ completely. This is made explicit in Theorem 3.11. Note the denominator factors to $\mathrm{Sp}(2n,\mathbb{R})$.

Proof: Let Θ_{DS} (resp. $\hat{\Theta}_{DS}$) be the character of π_{DS} (resp. $\hat{\pi}_{DS}$). By Theorem 2.3, (3.6) is equivalent to

$$\hat{\Theta}_{DS}(g) = n!(-1)^q \frac{\sum_{w\in W} \operatorname{sgn}(w)e^{w\lambda_0}}{\prod_{\alpha\in\Delta^+(H)}(e^{\alpha/2} - e^{-\alpha/2})}(g). \tag{3.7}$$

This follows more or less immediately from [12], [19], and the definition of coherent continuation.

We first show for g in the smaller set (cf. 3.1):

$$|e^{\beta_i}(g)| < 1 \quad (i = 1, \ldots, m), \tag{3.8}$$

$$\Theta_{DS}(g) = (-1)^q \frac{\sum_{W_K} \operatorname{sgn}(w)e^{w\lambda_0}}{\prod_{\alpha\in\Delta^+(H)}(e^{\alpha/2} - e^{-\alpha/2})}(g). \tag{3.9}$$

By [12] this holds with λ_0 replaced by $\rho = \lambda_0 + (\frac{1}{2}, \ldots, \frac{1}{2})$, and Θ_{DS} replaced by the holomorphic discrete series representation with infinitesimal character ρ. Then (3.9) follows by deforming ρ to λ as in ([19], Corollary 3.8). By the version of coherent continuation of [6] this implies (3.7), still for g in the set (3.8).

Now suppose $|e^{\beta_i}(g)| > 1$ for some i. Let $s_i = s_{\beta_i} \in W(\mathfrak{sp}, \mathfrak{h})$. Also write s_i for the corresponding reflection in $W(\mathfrak{sp}, \mathfrak{t})$, i.e. reflection in the root $c^*\beta_i \in \mathfrak{t}^*$. Write $g = \widetilde{\exp}(Z)\exp(X)$ as in (3.3)(a). It follows from a calculation in $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ that g is conjugate via $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ to $g' = \widetilde{\exp}(s_iZ)\exp(s_iX)$. This calculation is entirely in the covering group of $\mathrm{GL}(n, \mathbb{R})$, on which the cocycle has the simple form ([11], Corollary 5.5(2)). The right hand side of (3.7) is invariant upon replacing g by g'. It follows that (3.7) holds without condition (3.8), and this completes the proof.

The discussion at the end of the proof can be extended to consider reflections s_i in the long roots γ_i. It is easy to see that $\widetilde{\exp}(Z)\exp(X)$ $(Z \in \mathfrak{t}_0, X \in \mathfrak{a}_0)$ is conjugate via $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ to $\widetilde{\exp}(Z)\exp(s_iX)$, rather than $\widetilde{\exp}(s_iZ)\exp(s_iX)$. This reduces to the split Cartan subgroup $\tilde{H} = \tilde{T}_HA$ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and follows from the fact that $\tilde{T}_H \simeq \mathbb{Z}/4\mathbb{Z}$ is the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Furthermore, for any $\tau \in \lambda_0 + \Lambda$

$$e^\tau(\widetilde{\exp}(Z))/e^\tau(\widetilde{\exp}(s_iZ)) = e^{\gamma_i/2}(\widetilde{\exp}(Z)) = \pm 1. \tag{3.10}$$

We state the resulting formula, at the same time treating the case of $\omega(\psi_-)$.

THEOREM 3.11:

$$[\theta(\psi_\pm)_{\text{even}} - \theta(\psi_\pm)_{\text{odd}}](g) = \prod_{i=1}^{s} \operatorname{sgn}(1 + e^{\pm\gamma_i/2}(g)) \frac{\sum_{w\in W} \operatorname{sgn}(w) e^{w\lambda_0}}{\prod_{\alpha\in\Delta^+(H)}(e^{\alpha/2} - e^{-\alpha/2})}(g)$$
$$= \prod_{i=1}^{s} \operatorname{sgn}(1 + e^{\pm\gamma_i/2}(g)) \frac{1}{\prod_{i=1}^{n}(e^{\alpha_i/4} + e^{-\alpha_i/4})(g)}.$$

Proof: The first line for ψ_+ follows from the preceding discussion. With * denoting contragredient, $\theta(\psi_-)(g) = \theta(\psi_+)^*(g) = \theta(\psi_+)(g^{-1})$. The quotient $\frac{\sum_w}{\prod_\alpha}$ is invariant under $g \to g^{-1}$, and $e^{\gamma_i/2}(g^{-1}) = e^{-\gamma_i/2}(g)$. The second line follows exactly as in the introduction (cf. 1.5); there is no problem making this argument precise.

Remark 3.12: The terms $\operatorname{sgn}(1+e^{\pm\gamma_i/2}(g))$ are constant on each connected component of a Cartan subgroup, and on the identity component of each Cartan subgroup are independent of $\psi_\pm$. In particular, the restriction of $\theta(\psi)_{\text{even}} - \theta(\psi)_{\text{odd}}$ to a neighborhood of the identity in $\widetilde{\operatorname{Sp}}(2n,\mathbb{R})$ is independent of ψ.

We now make explicit choices of Cayley transforms and Cartan subgroups. Our discussion, as well as some notation, is similar to but not exactly the same as [14].

For non-negative integers m, r, s with $2m + r + s = n$ we define a Cartan subgroup $H^{m,r,s}$ of $\operatorname{Sp}(2n,\mathbb{R})$ with Lie algebra $\mathfrak{h}_0^{m,r,s}$, and a Cayley transform $c\colon \mathfrak{t} \simeq \mathfrak{h}^{m,r,s}$. Write $W = \mathbb{R}^{2n} = W_1 \oplus W_2 \oplus W_3$ where W_1 is spanned by $\{e_i, f_i | 1 \le i \le 2m\}$, W_2 by $\{e_i, f_i | 2m+1 \le i < 2m+r\}$ and W_3 by $\{e_i, f_j | 2m+r+1 \le i \le n\}$. We identify $\operatorname{Sp}(W_i)$ and $\mathfrak{sp}(W_i)$ with their images in $\operatorname{Sp}(2n,\mathbb{R})$ and $\mathfrak{sp}(2n,\mathbb{R})$. For $z_i = x_i + iy_i \in \mathbb{C}$, $1 \le i \le m$ let

$$\text{(3.13)(a)} \qquad \mathfrak{h}^{m,0,0}(z_1,\ldots,z_m) = \begin{pmatrix} & X & Y & \\ X & & & -Y \\ -Y & & & -X \\ & Y & -X & \end{pmatrix} \in \mathfrak{sp}(W_1)$$

where $X = \operatorname{diag}(x_1,\ldots,x_m)$ and $Y = \operatorname{diag}(y_1,\ldots,y_m)$. For $\theta_i \in \mathbb{R}$ $(1 \le i \le r)$ we let

$$\text{(b)} \qquad \mathfrak{h}^{0,r,0}(\theta_1,\ldots,\theta_r) = \begin{pmatrix} & X \\ -X & \end{pmatrix} \in \mathfrak{sp}(W_2)$$

with $X = \operatorname{diag}(\theta_1,\ldots,\theta_r)$, and for $x_i \in \mathbb{R}$ $(1 \le i \le s)$ let

$$\text{(c)} \qquad \mathfrak{h}^{0,0,s}(x_1,\ldots,x_s) = \operatorname{diag}(x_1,\ldots,x_s,-x_1,\ldots,-x_s) \in \mathfrak{sp}(W_3).$$

Taking the sum of these elements gives us an element

$$\text{(d)} \qquad \mathfrak{h}^{m,r,s}(z_1,\ldots,z_m,\theta_1,\ldots,\theta_r,x_1,\ldots,x_s)\in\mathfrak{sp}(2n,\mathbb{R})$$

and this defines a Cartan subalgebra $\mathfrak{h}_0^{m,r,s}$ of $\mathfrak{sp}(2n,\mathbb{R})$. The compact Cartan subalgebra is $\mathfrak{t}_0=\mathfrak{h}_0^{0,n,0}$. The complexification $\mathfrak{h}^{m,r,s}$ is defined in the obvious way.

Let $H^{m,r,s}\simeq\mathbb{C}^{*m}\times S^{1r}\times R^{*s}$ be the Cartan subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ with Lie algebra $\mathfrak{h}_0^{m,r,s}$, with inverse image $\tilde{H}^{m,r,s}$ in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. These are representatives for the conjugacy classes of Cartan subgroups of $\mathrm{Sp}(2n,\mathbb{R})$ and $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ respectively. The compact Cartan subgroup is $T=H^{0,n,0}$.

Given $H^{m,r,s}$ let

$$c_1=\frac{\sqrt{2}}{2}\begin{pmatrix} I_{2m} & -iI'_{2m}\\ -iI'_{2m} & I_{2m}\end{pmatrix}$$

where $I'_{2m}=\mathrm{antidiag}(1,\ldots,1)$, let $c_2=\frac{\sqrt{2}}{2}\begin{pmatrix} I_r & -iI_r\\ -iI_r & I_r\end{pmatrix}$ and let $c=c_1\times c_2\times I_s\in\mathrm{Sp}(W_{1\mathbb{C}})\times\mathrm{Sp}(W_{2\mathbb{C}})\times\mathrm{Sp}(W_3)\subset\mathrm{Sp}(2n,\mathbb{C})$. Then c is a Cayley transform for $H^{m,r,s}$, i.e. $\mathrm{ad}(c)(\mathfrak{t})=\mathfrak{h}^{m,r,s}$.

Recall $\Delta^+(H^{m,r,s})=c^{*-1}(\Delta^+(T))$. For $i=1,\ldots,n$ let $\alpha_i=c^{*-1}(2e_i)$; these are the long roots of $\Delta^+(H^{m,r,s})$. The strongly orthogonal real roots β_i,γ_i (cf. 3.1) are $\beta_i=c^{-1}(e_{2i-1}+e_{2i})$ $(i=1,\ldots m)$ and $\gamma_i=c^{-1}(2e_{2m+r+i})$ $(i=1,\ldots,s)$. For later use we compute the Weyl reflections corresponding to β_i,γ_i. Write X as in (3.13)(d). Then $\beta_i(X)=z_i+\overline{z}_i$ and $\gamma_i(X)=2x_i$. Therefore

$$\text{(3.14)(a)} \qquad s_{\beta_i}(X)=\mathfrak{h}^{m,r,s}(z_1,\ldots,-\overline{z}_i,\ldots,z_m,\theta_1,\ldots,\theta_r,x_1,\ldots,x_s)$$

and

$$\text{(b)} \qquad s_{\gamma_i}(X)=\mathfrak{h}^{m,r,s}(z_1,\ldots,z_m,\theta_1,\ldots,\theta_r,x_1,\ldots,-x_i,\ldots x_s).$$

We make the decomposition (3.3)(a) more explicit. Write $g\in\tilde{H}^{m,r,s}$ as

$$\text{(3.15)(a)} \qquad g=t\cdot\exp(X)=\widetilde{\exp}(Z)\exp(X).$$

Here

$$\text{(b)} \qquad X=\mathfrak{h}_0^{m,r,s}(x_1,\ldots,x_m,\overbrace{0,\ldots,0}^{r},c_1,\ldots,c_s)\in\mathfrak{a}_0 \quad (x_i,c_j\in\mathbb{R})$$

and

$$\text{(c)}\quad Z = \mathfrak{h}^{0,n,0}(y_1, -y_1, \ldots, y_m, -y_m, \theta_1, \ldots, \theta_r, \pi k_1, \ldots, \pi k_s) + \frac{1-\delta}{2}\mathfrak{h}^{0,n,0}(2\pi, 0, \ldots, 0)$$

for some $y_i, \theta_j \in \mathbb{R}$, $k_i \in \mathbb{Z}$ and $\delta = \pm 1$. The parameter δ has been chosen for later convenience (cf. Proposition 4.5).

Remark 3.16: The choice of Z is not at all unique. If $r + s \neq 0$ we may assume $\delta = 1$ (by modifying the first term of Z if necessary). Only in the case $2m = n, r = s = 0$ is it necessary to take $\delta = -1$. In this case we can replace this term with the more natural

$$Z' = \mathfrak{h}^{0,n,0}(\pi, \ldots, \pi, (-1)^{m+1}\pi);$$

note that $\widetilde{\exp}(Z')$ is a central element of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ lying over $-I \in \mathrm{Sp}(2n, \mathbb{R})$.

In these coordinates

$$\text{(3.17)}\qquad \mathrm{sgn}(1 + e^{\pm\gamma_i}(g)) = (\mp \mathrm{sgn}(c_i))^{k_i}$$

(cf. Theorem 3.11).

We compute the character of the oscillator representation in these coordinates. Recall $p(w) \in W_K$ was defined in (2.8)(b). This also follows from ([14], §1, 7.8).

PROPOSITION 3.18: *Write $g = t \cdot \exp(X)$ as in (3.15).*

$$\begin{aligned}[\theta(\psi_\pm)_{\mathrm{even}} &+ \theta(\psi_\pm)_{\mathrm{odd}}](g) \\ &= (\mp 1)^n \prod_{i=1}^{s} (\mp \mathrm{sgn}(c_i))^{k_i+1} \frac{\sum_{w\in W} \mathrm{sgn}(p(w)) e^{w\lambda_0}}{\prod_{\alpha\in\Delta^+(H^{m,r,s})}(e^{\alpha/2} - e^{-\alpha/2})}(g) \\ &= (\mp 1)^n \prod_{i=1}^{s} (\mp \mathrm{sgn}(c_i))^{k_i+1} \frac{1}{\prod_{i=1}^{n}(e^{\alpha_i/4} - e^{-\alpha_i/4})(g)}.\end{aligned}$$

Proof: Let $Z_0 = \mathfrak{h}^{0,n,0}(\pi, \ldots, \pi)$ and let $z = \widetilde{\exp}(Z_0)$. This is a central element of $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ lying over $-I \in \mathrm{Sp}(2n, \mathbb{R})$. Then $\theta(\psi_\pm)_{\mathrm{even}}(z) = i^{\pm n}$, as is seen by considering the lowest K-type $\pm(\frac{1}{2}, \ldots, \frac{1}{2})$ of $\theta(\psi_\pm)_{\mathrm{even}}$. On the other hand $\theta(\psi_\pm)_{\mathrm{odd}}(z) = -i^{\pm n}$. Therefore

$$\text{(3.19)}\qquad [\theta(\psi_\pm)_{\mathrm{even}} + \theta(\psi_\pm)_{\mathrm{odd}}](g) = i^{\mp n}[\theta(\psi_\pm)_{\mathrm{even}} - \theta(\psi_\pm)_{\mathrm{odd}}](zg).$$

We compute the right hand side by Theorem 3.11. Replacing g by zg replaces Z (3.15)(c) by $Z + Z_0$, hence k_i is replaced by $k_i + 1$. This accounts for the $(\mp \operatorname{sgn}(c_i))$ term. Each term $e^{w\lambda_0}(g)$ is multiplied by $e^{w\lambda_0}(Z_0)$. This is readily computed to be $i^{n^2}(-1)^{t(w)}$ where $t(w)$ is the number of sign changes in w. Note that $(-1)^{t(w)} = \operatorname{sgn}(w)\operatorname{sgn}(p(w))$ with p as in (2.8)(b). Finally each non-compact root term in the Weyl denominator changes sign, giving a factor of $(-1)^q$. The constant is therefore $i^{\mp n} i^{n^2}(-1)^q = (\mp 1)^n$. This proves the first statement. The second follows from the fact, left as an exercise, that the numerator equals $\prod_\alpha (e^{\alpha/2} - e^{-\alpha/2}) \prod_i (e^{\alpha_i/4} + e^{-\alpha_i/4})$, where the first product is over the short positive roots. This completes the proof.

4. Determinants

We state as lemmas some straightforward calculations on the compact and split Cartan subgroups of $\mathrm{SL}(2,\mathbb{R})$, and the complex Cartan subgroup of $\mathrm{Sp}(4,\mathbb{R})$. The proofs are left to the reader.

Definition 4.1: For $\theta \in \mathbb{R}$ let

$$\zeta(\theta) = \begin{cases} \operatorname{sgn}\cos(\theta) & \cos(\theta) \neq 0, \\ \sin(\theta) & \cos(\theta) = 0. \end{cases}$$

LEMMA 4.2: *For $\theta \in \mathbb{R}$ let $X_\theta = \mathfrak{h}^{0,1,0}(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, an element of the compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{sl}(2,\mathbb{R})$, and let $\tilde{t}_\theta = \widetilde{\exp}(X_\theta) \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$. For α a root of $\mathfrak{t}$,*

(a)
$$\frac{1}{e^{\alpha/4} + e^{-\alpha/4}}(\tilde{t}_\theta) = \frac{1}{2\cos(\theta/2)}.$$

Let $t_\theta = p(\tilde{t}_\theta) \in \mathrm{SL}(2,\mathbb{R})$. Then

(b)
$$\det(1 + t_\theta) = 4\cos^2(\theta/2) \geq 0$$

and

(c)
$$\sqrt{\det(1 + t_\theta)} = |\det(1 + t_\theta)|^{\frac{1}{2}} = 2\cos(\theta/2)\zeta(\theta/2).$$

LEMMA 4.3: *For $c_0 \in \mathbb{R}$ let $X = \mathfrak{h}^{0,0,1}(c_0) = \mathrm{diag}(c_0, -c_0)$, an element of the split Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{sl}(2,\mathbb{R})$. For $k \in \mathbb{Z}$ let $Z = \mathfrak{h}^{0,1,0}(\pi k) \in \mathfrak{t}_0$, and write $c = c_0 + \pi i k$. Let $\tilde{a}_c = \widetilde{\exp}(Z)\exp(X) \in \tilde{A} = \tilde{H}^{0,0,1} \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$.*

For α a root of $\mathfrak{a}$,

$$\text{(a)} \qquad \frac{1}{e^{\alpha/4}+e^{-\alpha/4}}(\tilde{a}_c) = \frac{1}{2\cosh(c/2)}.$$

Let $a_c = p(\tilde{a}_c) = \mathrm{diag}(e^c, e^{-c}) \in \mathrm{SL}(2,\mathbb{R})$. Then

$$\text{(b)} \qquad \det(1+a_c) = (2\cosh(c/2))^2$$

and

$$\text{(c)} \qquad |\det(1+a_c)|^{\frac{1}{2}} = i^k(-\mathrm{sgn}(c_0))^k 2\cosh(c/2).$$

LEMMA 4.4: *For $z \in \mathbb{C}$ let $Z = \mathfrak{h}^{1,0,0}(z) \in \mathfrak{sp}(4,\mathbb{R})$. For $\delta = \pm 1$ let Z_0 be the element $\frac{1-\delta}{2}\mathfrak{h}^{0,2,0}(\pi,\pi)$ of the compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{sp}(4,\mathbb{R})$ (cf. Remark 3.16). Let $\tilde{g} = \widetilde{\exp}(Z_0)\widetilde{\exp}(Z) \in \widetilde{\mathrm{Sp}}(4,\mathbb{R})$. With $\pm\beta_1, \pm\beta_2$ the long roots of $\mathfrak{h}^{1,0,0}$,*

$$\text{(a)} \qquad \frac{1}{(e^{\beta_1/4}+e^{-\beta_1/4})(e^{\beta_2/4}+e^{-\beta_2/4})}(\tilde{g}) = \frac{\delta}{2(\cosh(x)+\delta\cos(y))}.$$

Let $g = p(\tilde{g}) \in \mathrm{Sp}(4,\mathbb{R})$. Then

$$\text{(b)} \qquad \det(1+g) = [2(\cosh(x)+\delta\cos(y))]^2 \geq 0$$

and

$$\text{(c)} \qquad \sqrt{\det(1+g)} = |\det(1+g)|^{\frac{1}{2}} = 2(\cosh(x)+\delta\cos(y)).$$

PROPOSITION 4.5: *With notation as in Corollary 3.18,*

$$[\theta(\psi_\pm)_{\mathrm{even}} - \theta(\psi_\pm)_{\mathrm{odd}}](\tilde{g}) = \frac{\prod_{i=1}^r \zeta(\theta_i/2)\prod_{j=1}^s (\pm i)^{k_j}\delta}{|\det(1+g)|^{\frac{1}{2}}}.$$

Proof: This follows immediately from Theorem 3.11 and Lemmas 4.2–4.4.

Recall (Remark 3.16) we may assume $\delta = 1$ except in the case $2m = n, r = s = 0$.

For the definition and properties of the Weil invariant $\gamma(x,\psi)$ we refer to ([11], Appendix). For $x \in \mathbb{R}^*$ let $\sqrt[\psi]{x} = \gamma(x,\psi)\sqrt{|x|}$. This is a choice of square root of x and satisfies $\sqrt[\psi]{xy} = \sqrt[\psi]{x}\,\sqrt[\psi]{y}(x,y)_{\mathbb{R}}$ where $(\,,)_{\mathbb{R}}$ is the Hilbert symbol [13]. Also $\gamma(x,\psi_a) = \gamma(x,\psi_{ab^2})$ for all $b \in \mathbb{R}^*$, and $\gamma(x,\psi_\pm) = 1$ $(x>0)$ or $\mp i$ $(x<0)$.

We express the preceding result in terms of $\sqrt[\psi]{\det(1+g)}$. In Lemmas 4.2 and 4.4, $\sqrt[\psi]{\det(1+g)} = |\det(1+g)|^{\frac{1}{2}}$, whereas in Lemma 4.3 a short calculation gives

$$\sqrt[\psi]{\det(1+g)} = \gamma((-1)^k,\psi)|\det(1+g)|^{\frac{1}{2}}.$$

Noting that $\gamma((-1)^k,\psi_\pm) = (\mp i)^k(-1)^{k(k-1)/2}$ gives the next result.

PROPOSITION 4.6: *With notation as in Proposition* 3.18,

$$[\theta(\psi)_{\mathrm{even}} - \theta(\psi)_{\mathrm{odd}}](\tilde{g}) = \frac{\prod_{i=1}^{r}\zeta(\theta_i/2)\prod_{j=1}^{s}(-1)^{k_j(k_j-1)/2}\delta}{\sqrt[\psi]{\det(1+g)}}.$$

Note that the numerator is independent of ψ, and the role of ψ is solely in the branch of the square root. Finally we proceed as in the proof of Proposition 3.18 to obtain a result for $\theta(\psi_\pm)_{\mathrm{even}} + \theta(\psi_\pm)_{\mathrm{odd}}$ (cf. [14], §I, Théorème 2).

PROPOSITION 4.7: *With notation as in Proposition* 3.18,

$$[\theta(\psi_\pm)_{\mathrm{even}} + \theta(\psi_\pm)_{\mathrm{odd}}](\tilde{g}) = (\pm i)^r \frac{\prod_{i=1}^{r}\mathrm{sgn}(\sin(\theta_i/2))\prod_{j=1}^{s}(\pm i)^{k_j}\delta}{|\det(1-g)|^{\frac{1}{2}}}.$$

5. Cocycles

In this section we write the Cartan subgroup $\tilde{H}^{m,r,s}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ in terms of a certain cocycle. Proposition 4.6 then has a very simple form.

Let $\tilde{c}_n(\,,)$ be the "Rao" cocycle on $\mathrm{Sp}(2n,\mathbb{R})$, i.e. the normalized ±1-valued cocycle of [11]. Henceforth we let $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ be the metaplectic cover of $\mathrm{Sp}(2n,\mathbb{R})$ realized explicitly via the Rao cocycle, i.e. as pairs $(g;\epsilon)$ $(g \in \mathrm{Sp}(2n,\mathbb{R}), \epsilon = \pm1)$ with multiplication $(g,\epsilon)(g',\epsilon') = (gg',\epsilon\epsilon'\tilde{c}_n(g,g'))$.

From the preceding results it is clear the calculation of the character of the oscillator representation restricted to a Cartan subgroup $\tilde{H}^{m,r,s}$ comes down to a calculation on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $\widetilde{\mathrm{Sp}}(4,\mathbb{R})$. We have written the Cartan subgroup $\tilde{H}^{m,r,s}$ of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ in terms of the exponential map $\widetilde{\exp}$. To express our results in terms of the Rao cocycle, it is necessary to write $\widetilde{\exp}$ in these terms. It turns out to be simpler to write the cover $\tilde{H}^{m,r,s}$ in terms of a closely related cocycle $\hat{c}_H$. The calculation of $\widetilde{\exp}$ reduces as well to $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Sp}(4,\mathbb{R})$.

Let x: $\mathrm{Sp}(2n,\mathbb{R}) \to \pm 1$ be the map of ([11], Lemma 5.1). Recall some notation from Lemma 4.2: $t_\theta = \exp(X_\theta)$ is an element of the compact Cartan subgroup T of $\mathrm{SL}(2,\mathbb{R})$. For $x \in \mathbb{R}^*$ let $\alpha_x = \mathrm{diag}(x, \frac{1}{x})$, an element of the split Cartan subgroup A of $\mathrm{SL}(2,\mathbb{R})$. In the notation of Lemma 4.3, $a_c = \alpha_{e^c}$. Then $x(\alpha_y) = \mathrm{sgn}(y)$, and

$$x(t_\theta) = \begin{cases} -\mathrm{sgn}(\sin(\theta)) & \sin(\theta) \neq 0, \\ \cos(\theta) & \sin(\theta) = 0. \end{cases} \tag{5.1}$$

Recall ζ was defined in Section 4 (Definition 4.1).

LEMMA 5.2: *For $t_\theta, t_\phi \in T$,*

$$\tilde{c}_1(t_\theta, t_\phi) = \zeta\left(\frac{\theta}{2}\right)\zeta\left(\frac{\phi}{2}\right)\zeta\left(\frac{\theta+\phi}{2}\right) \tag{5.2)(a}$$

and

$$\widetilde{\exp}(X_\theta) = \left(t_\theta; \zeta\left(\frac{\theta}{2}\right)\right). \tag{b}$$

For $\alpha_x, \alpha_y \in A$,

$$\tilde{c}_1(\alpha_x, \alpha_y) = (x, y)_{\mathbb{R}}. \tag{c}$$

In the notation of Lemma 4.3, let $\tilde{a}_c$ $(c = c_0 + \pi i k)$ be an element of $\tilde{A}$, and let $a_c = p(\tilde{a}_c) = \alpha_{e^c}$. Then

$$\tilde{a}_c = (a_c; (-1)^{k(k-1)/2}). \tag{d}$$

The restriction of the cocycle $\tilde{c}_2(,)$ to the complex Cartan subgroup $H^{1,0,0}$ of $\mathrm{Sp}(4,\mathbb{R})$ *is trivial. Let $\tilde{g} = \tilde{t} \cdot \exp(X)$ as in Lemma 4.4, with image g in $\tilde{H}^{1,0,0}$. Then*

$$\tilde{g} = (g; \delta) \tag{e}$$

Proof: By [11] (the Remark following Corollary 5.8),

$$\tilde{c}_1(t_\theta, t_\phi) = (x(\theta), x(\phi))_{\mathbb{R}}(-x(\theta)x(\phi), x(\theta+\phi))_{\mathbb{R}} \tag{5.3)(a}$$

where we have written $x(\theta) = x(t_\theta)$. Statement (a) follows from a straightforward but tedious calculation. It is easier, however, to proceed by an indirect method which has the advantage of proving (a) and (b) simultaneously.

Write $\widetilde{\exp}(X_\theta) = (t_\theta, \psi(\theta))$ for some ψ: $\mathbb{R} \to \pm 1$. To prove (b) we need to show $\psi(\theta) = \zeta(\frac{\theta}{2})$. The condition that $\widetilde{\exp}$ is a group homomorphism from $\mathfrak{t}_0$ to $\tilde{T}$ gives

$$\text{(b)} \qquad \psi(\theta)\psi(\phi) = \tilde{c}_1(t_\theta, t_\phi)\psi(\theta + \phi)$$

for all θ, ϕ. Set $\theta = \phi$ to conclude

$$\text{(c)} \qquad \psi(\theta) = \tilde{c}_1(t_{\frac{\theta}{2}}, t_{\frac{\theta}{2}})$$

which by (a) and elementary properties of the Hilbert symbol [13] equals

$$\text{(d)} \qquad \left(x\left(\frac{\theta}{2}\right) x(\theta)\right).$$

Write $e^{i\frac{\theta}{2}} = x + iy$. For generic θ we conclude

$$\text{(e)} \qquad \psi(\theta) = \operatorname{sgn}((-y)(-2xy)) = \operatorname{sgn}(x) = \operatorname{sgn}\cos\left(\frac{\theta}{2}\right) = \zeta\left(\frac{\theta}{2}\right)$$

proving (5.2)(b) for generic θ.

This argument holds provided $xy \neq 0$. If $y = 0$ (5.3)(d) gives $\psi(\theta) = x(x^2 - y^2) = x = \cos(\frac{\theta}{2}) = \zeta(\frac{\theta}{2})$, whereas $x = 0$ gives $\psi(\theta) = (-y)(x^2 - y^2) = y = \sin(\frac{\theta}{2}) = \zeta(\frac{\theta}{2})$. This proves (5.2)(b), and (a) follows from (5.3)(b).

Statement (5.2)(c) and the triviality of the cocycle on $H^{1,0,0}$ follow immediately from ([11], Corollary 5.5(2)).

Statements (5.2)(d) and (e) follow from (b), since the covering is essentially on T. Thus with $\tilde{a}_c = \widetilde{\exp}(\pi k)\exp(c_0)$ as usual, (b) implies $\tilde{a}_c = (a_c; \zeta(\frac{\pi k}{2}))$, and then (d) follows from the definition of ζ. Finally if $\delta = 1$, (e) is immediate since the cocycle is trivial and $H^{1,0,0}$ is connected. For $\delta = -1$ it follows from the fact that $\widetilde{\exp}(\mathfrak{h}^{0,1,0}(\pi, \pi)) = (-I; -1)$. This completes the proof.

Remark 5.4: Note that although $\zeta(\frac{\theta}{2})$ is not well defined on T, the product of the three terms on the right hand side of (5.2)(a) is well defined. If $\theta \to \zeta(\frac{\theta}{2})$ were well-defined on T, (5.2)(a) would say c is a co-boundary, which of course is not the case. The cocycle on T may be lifted to the universal cover $\mathbb{R} \overset{\exp}{\to} T$, and (5.2)(a) can be interpreted as an expression for this lift as a coboundary, pushed down to T.

It is an immediate consequence of (5.2)(b,d,e) and Proposition 4.6 that on the compact or split Cartan subgroups of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ or on the complex Cartan subgroup $\tilde{H}^{1,0,0}$ of $\widetilde{\mathrm{Sp}}(4,\mathbb{R})$ we have the simple formula

$$[\theta(\psi)_{\text{even}} - \theta(\psi)_{\text{odd}}](g;\epsilon) = \frac{\epsilon}{\sqrt[4]{\det(1+g)}}. \tag{5.5}$$

With the appropriate cocycle on $H^{m,r,s}$, (5.5) extends to $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. As noted earlier, the calculations of the precedings sections reduce to $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Sp}(4,\mathbb{R})$, ultimately because this is true of the second formula in Theorem 3.11. The natural cocycle on $H^{m,r,s}$ is therefore "diagonal".

Recall (Section 3) $H^{m,r,s}$ is isomorphic to $\mathbb{C}^{*m} \times S^{1r} \times \mathbb{R}^{*s}$, and we introduce coordinates on $H^{m,r,s}$ accordingly: for $z_i \in \mathbb{C}^*, u_i \in S^1$ and $x_i \in \mathbb{R}^*$, we let

$$H^{m,r,s}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s) \tag{5.6)(a}$$

be the corresponding element of $H^{m,r,s}$. More precisely, for $w_i \in \mathbb{C}, \theta_i \in \mathbb{R}$ and $c_i \in \mathbb{R} + \pi i Z$,

$$H^{m,r,s}(e^{w_1},\dots,e^{w_m},e^{i\theta_1},\dots,e^{i\theta_r},e^{c_1},\dots,e^{c_s}) \tag{5.6)(b}$$

is the exponential of the element

$$\mathfrak{h}^{m,r,s}(w_1,\dots,w_m,\theta_1,\dots,\theta_r,c_1,\dots,c_s) \tag{5.6)(c}$$

of the complex Cartan subalgebra $\mathfrak{h}^{m,r,s}$.

Definition 5.7: Let $H^{m,r,s} \simeq \mathbb{C}^{*m} \times S^{1r} \times \mathbb{R}^{*s}$ be one of our chosen Cartan subgroups of $\mathrm{Sp}(2n,\mathbb{R})$. We define a cocycle $\hat{c}_H(,)$ on H to be the product of the cocycles on each factor obtained by restriction from $\mathrm{SL}(2,\mathbb{R})$ (to $S^1, \mathbb{R}^*$) and $\mathrm{Sp}(4,\mathbb{R})$ (to $\mathbb{C}^*$). Define $\hat{H}^{m,r,s}$ to be the two-fold cover of $H^{m,r,s}$ defined by $\hat{c}_H(,)$.

To be precise, suppose

$$g = H^{m,r,s}(z_1,\dots,z_m,e^{i\theta_1},\dots,e^{i\theta_r},x_1,\dots,x_s)$$

and similarly g'. Then

$$\hat{c}_H(g,g') = \prod_{i=1}^{r} \tilde{c}_1(t_{\theta_i},t_{\theta_i'}) \prod_{i=1}^{s} \tilde{c}_1(\alpha_{x_i},\alpha_{x_i'})$$

with $\tilde{c}_1(,)$ given explicitly in Lemma 5.2.

We will see $\hat{H}^{m,r,s}$ is isomorphic to $\tilde{H}^{m,r,s}$ (Lemma 5.9).

Definition 5.8: Write $g \in H^{m,r,s}$ as in (5.6)(a–c). For $1 \le i \le r+s$ let

$$x_i(g) = \begin{cases} x(t_{\theta_i}), & j = 1, \dots, r, \\ x(\alpha_{x_{j-r}}), & j = r+1, \dots, r+s. \end{cases}$$

Define

$$\tau(g) = \prod_{1 \le i < j \le r+s} (x_i(g), x_j(g))_{\mathbb{R}}.$$

Finally define a map ϕ: $\hat{H}^{m,r,s} \to \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ by

$$\phi(g; \epsilon) = (g; \epsilon\tau(g)).$$

The image of ϕ is $\tilde{H}^{m,r,s}$.

LEMMA 5.9:

(1) *For $g, h \in H^{m,r,s}$,*

$$\tilde{c}_n(g, h) = \hat{c}_H(g, h)\tau(g)\tau(h)\tau(gh),$$

in other words $\tilde{c}_n(,)$ restricted to $H^{m,r,s}$ and $\hat{c}_H$ differ by the coboundary of τ.

(2) *ϕ is an isomorphism between $\hat{H}^{m,r,s}$ and $\tilde{H}^{m,r,s}$.*

(3) *Write $\tilde{g} = \tilde{t} \cdot \exp(X) = \widetilde{\exp}(Z)\exp(X) \in \tilde{H}^{m,r,s}$ as in (3.15)(a–c), and let $g = p(\tilde{g})$. Then*

$$\phi^{-1}(\tilde{g}) = (g; \epsilon) \tag{5.10)(a}$$

with

$$\epsilon = \prod_{i=1}^{r} \zeta\left(\frac{\theta_i}{2}\right) \prod_{i=1}^{s} (-1)^{k_i(k_i-1)/2}\delta. \tag{5.10)(b}$$

Proof: We consider (1). If $H^{m,r,s}$ has only one factor, i.e. $H^{m,r,s} \simeq \mathbb{C}^*, S^1$ or $\mathbb{R}^*$, this is precisely how $\hat{c}_H(,)$ was defined. In the case of two factors (1) becomes ([11], Corollary 5.6). The general case follows by induction on the number of factors. Statement (2) is an immediate consequence of (1), and (3) follows from (5.2)(b,d,e) and the fact that the cocycle $\hat{c}_H$ is diagonal.

Via ϕ we identify $\tilde{H}^{m,r,s}$ and $\hat{H}^{m,r,s}$, and write the character of the oscillator representation as a function on the regular elements of $\hat{H}^{m,r,s}$.

PROPOSITION 5.11: *Let $\tilde{g} = (g;\epsilon)$ be an element of $\hat{H}^{m,r,s}$, with g a regular element of $H^{m,r,s}$. Then*

$$[\theta(\psi)_{\text{even}} - \theta(\psi)_{\text{odd}}](g;\epsilon) = \frac{\epsilon}{\sqrt[\psi]{\det(1+g)}}.$$

Proof: This follows immediately from Proposition 4.6 and Lemma 5.9.

The following formula directly on $\tilde{H}^{m,r,s} \subset \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (without recourse to $\hat{H}^{m,r,s}$) follows immediately from Lemma 5.9.

COROLLARY 5.12: *Let $(g;\epsilon)$ be a regular element of $\tilde{H}^{m,r,s} \subset \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Then*

$$[\theta(\psi)_{\text{even}} - \theta(\psi)_{\text{odd}}](g;\epsilon) = \frac{\epsilon\tau(g)}{\sqrt[\psi]{\det(1+g)}}.$$

6. The complex case

The corresponding results for the oscillator representation of the complex group $G = \mathrm{Sp}(2n,\mathbb{C})$ are much easier, and easily read off from [14]. For the sake of completeness we include the statement.

Write $\omega = \omega_{\text{even}} \oplus \omega_{\text{odd}}$ for the oscillator representation of G. This is unique up to isomorphism. Write $\theta = \theta_{\text{even}} + \theta_{\text{odd}}$ for its character.

We identify a Cartan subgroup H of G with C^{*n} as usual. We also identify $\mathfrak{h}_0 = \mathrm{Lie}(H)$ and $\mathfrak{h}_0^*$ with $\mathbb{C}^n$. For $\lambda_1, \lambda_2 \in \mathbb{C}^n$ satisfying $\lambda_1 - \lambda_2 \in 2\pi i\mathbb{Z}^n$ the standard module $X(\lambda_1,\lambda_2)$ [20] is defined. Let $\lambda = (n-\frac{1}{2},\dots,\frac{3}{2},\frac{1}{2})$. Then (λ,λ) is the infinitesimal character of the oscillator representation, and $X(\lambda,\lambda)$ contains ω_{even} as a constituent. The coherent continuation action of the Weyl group W of type C_n is defined on representations with this infinitesimal character, and $w\cdot X(\lambda,\lambda) = X(w\lambda,\lambda)$.

PROPOSITION 6.1:

$$\sum_{w\in W} w\cdot X(\lambda,\lambda) = \omega_{\text{even}} - \omega_{\text{odd}}.$$

This may be proved in the same way as Theorem 2.3. For another proof see ([14], §II.2). As an immediate consequence we conclude:

PROPOSITION 6.2: *For g a regular element of H,*

$$(\theta_{\text{even}} - \theta_{\text{odd}})(g) = \frac{\sum_{w\in W\times W}\mathrm{sgn}(w)e^{w(\lambda,\lambda)}}{\sum_{w\in W\times W}\mathrm{sgn}(w)e^{w(\rho,\rho)}}(g).$$

Finally, proceeding as in Section 4 we obtain:

PROPOSITION 6.3: *For g a regular element of H,*

$$(\theta_{\text{even}} - \theta_{\text{odd}})(g) = \frac{1}{|\det(1+g)|}.$$

Alternatively we may embed $\mathrm{Sp}(2n,\mathbb{C})$ in $\mathrm{Sp}(4n,\mathbb{R})$, taking $H \simeq \mathbb{C}^{*n}$ to $H^{n,0,0}$. Noting that the oscillator representation of $\widetilde{\mathrm{Sp}}(4n,\mathbb{R})$ restricts to the oscillator representation of $\mathrm{Sp}(2n,\mathbb{C})$, this follows from Proposition 4.5. This also follows by taking $-g$ in ([14], §II, Théorème 1).

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